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A Mathematical Model to Solve Nonlinear Initial and Boundary Value Problems by LDTM

Kiranta Kumari*, Praveen Kumar Gupta**, Gauree Shanker*
***Department of Mathematics & Statistics, Banasthali University, Banasthali-304022, Rajasthan
****Department of Mathematics, National Institute of Technology, Silchar, (An Institute of National Importance), Shilchar-788010, Assam

Abstract
In this paper, a novel method called Laplace-differential transform method (LDTM) is used to obtain an approximate analytical solution for strong nonlinear initial and boundary value problems associated in engineering phenomena. It is determined that the method works very well for the wide range of parameters and an excellent agreement is demonstrated and discussed between the approximate solution and the exact one in three examples. The most significant features of this method are its capability of handling non-linear boundary value problems.
Keywords: LDTM, Non-linear PDEs, Initial Conditions, Boundary conditions.

I. Introduction

Many mathematical and physical problems are described by ordinary or partial differential equations with appropriate initial or boundary conditions, these problems are usually formulated as initial value Problems or boundary value problems. In 1978, a Russian mathematician G.E. Pukhov proposed Differential transform method (DTM) and started from computational structure to solve differential equations by Taylor's transformation. But J. K. Zhou (1986) has proposed the algorithm and main application of DTM with linear and non-linear initial value problems in electric circuit analysis.

Ravi Kanth and Aruna (2009) have extended the DTM to solve the linear and non-linear Klein-Gordon equations and confirmed that the proposed technique provides its computational effectiveness and accuracy. Singh *et* al. (2010) have applied Homotopy perturbation transform method (HPTM) to obtain the solution of the linear and nonlinear Klein-Gordon equations. Khan *et* al. (2010) have given Laplace decomposition method to find the approximate solution of non-linear system of PDEs and found that the Laplace decomposition method and Adomian decomposition method both can be applied alternatively for the solution of higher order initial value problems. Jafari *et* al. (2010) have proposed two-dimensional differential transform method to solve nonlinear Gas Dynamic and Klein-Gordon equations. Haziqah *et* al. (2011) have studied higher-order boundary value problems (HOBVP) for higher-order nonlinear differential equation and make comparison among differential transformation method (DTM), Adomian decomposition method (ADM), and exact solutions. The researchers found that the DTM is more accurate than the Adomian Decomposition Method.

Alquran *et* al. (2012) have proposed a combined form of Laplace transform method and the DTM for solving non-homogeneous linear PDEs with variable coefficients and found that the LDTM requires less computational work compared to DTM. Mishra and Nagar (2012) have applied a coupling of Laplace transform method with Homotopy perturbation method which is called He-Laplace method to solve linear and non-linear PDEs and found that the technique is capable to reduce the volume of computational work as compared to Adomian polynomials. Hosseinzadeh and Salehpour (2013) have applied reduced differential transform method (RDTM) for finding approximate and exact solutions of some PDEs with variable coefficients.

In this paper, we use a combination of the Laplace transform method and the DTM for solving nonlinear PDEs with initial and boundary conditions. The goodness of this method is its capability of combining two strongest methods for finding fast convergent series solution of PDEs. The main aim of this paper is to solve nonlinear initial and boundary value problems by using Laplace-differential transform method. This paper considers the effectiveness of the Laplace-differential transform method for solving nonlinear equations. The rest of this paper is organized as follows: in section 2, the DTM is introduced. Section 3, is devoted to apply the basic idea of LDTM. Section 4, contains applications of LDTM. The conclusions are included in last Section.

II. Differential Transformation Method

The one variable differential transform [10] of a function $u(x,t)$ is defined as

$$U_k(t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0} \quad ; k \geq 0 \tag{2.1}$$

where $u(x,t)$ is the original function and $U_k(t)$ is the transformed function. The inverse of one variable differential transform of $U_k(t)$ is defined as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x - x_0)^k, \tag{2.2}$$

where x_0 is the initial point for the given condition. Then the function $u(x,t)$ can be written as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t) x^k. \tag{2.3}$$

III. Basic idea of LDTM

To illustrate the basic idea of Laplace differential transform method [7], we consider the general form of non-homogeneous PDEs with variable or constant coefficients

$$\pounds[u(x,t)] + \Re[u(x,t)] = f(x,t), \; x \in R, \; t \in R^+, \tag{3.1}$$

subject to the initial conditions

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \tag{3.2}$$

and the Dirichlet boundary conditions

$$u(0,t) = h_1(t), \; u(1,t) = h_2(t), \tag{3.3}$$

or the Neumann boundary conditions

$$u(0,t) = h_1(t), \; u_x(1,t) = h_3(t), \tag{3.4}$$

where $\pounds[.]$ is linear operator w.r.to 't', $\Re$ [.] is remaining operator and *f* is a known analytical function.
First, we take the Laplace transform on both sides of equation (3.1), w.r.to '*t*', and we get

$$\mathrm{L}[\pounds[u(x,t)]] + \mathrm{L}[\Re[u(x,t)]] = \mathrm{L}[f(x,t)]. \tag{3.5}$$

By using initial conditions from equation (3.2), we get

$$\bar{\mathrm{u}}(x,s) + \mathrm{L}[\Re[u(x,t)]] = \bar{f}(x,s),$$

where $\bar{\mathrm{u}}(x,s)$ and $\bar{f}(x,s)$ are the Laplace transform of $\mathrm{u}(x,t)$ and $f(x,t)$ respectively.
Afterwards, we apply differential transform method w. r. to '*x*', and we get

$$\overline{\mathrm{U}}_\mathrm{k}(s) + \mathrm{L}[\Re[U_k(t)]] = \overline{F}_k(s), \tag{3.6}$$

where $\overline{\mathrm{U}}_\mathrm{k}(s)$ and $\overline{\mathrm{F}}_\mathrm{k}(s)$ are the differential transform of $\bar{\mathrm{u}}(x,s)$ and $\bar{f}(x,s)$ respectively.
In the next step, we apply inverse Laplace transform on both sides w. r. to '*s*', and then we get

$$\mathrm{L}^{-1}\left[\overline{U}_k(s)\right] + \mathrm{L}^{-1}\mathrm{L}[\Re[U_k(t)]] = \mathrm{L}^{-1}\left[\overline{F}_k(s)\right],$$

or

$$U_k(t) + \Re[U_k(t)] = F_k(t). \tag{3.7}$$

Now, apply the differential transform method on the given Dirihlet and Neumann boundary conditions (3.3) and (3.4), we get

$$U_0(t) = h_1(t). \tag{3.8}$$

Let us assume

$$U_1(t) = a\,q(t). \tag{3.9}$$

By the definition of DTM [3], we take

$$u(1,t) = \sum_{i=0}^{\infty} U_i(t), \; u_x(1,t) = \sum_{i=0}^{\infty} i\,U_i(t). \tag{3.10}$$

By equation (3.10), we calculate the value of '*a*'.
Now by the above equations (3.8) and (3.9) in (3.7), the closed form series solution can be written as

$$u(x,t)=\sum_{k=0}^{\infty}U_k(t)x^k. \tag{3.11}$$

IV. Numerical Examples

To illustrate the applicability of LDTM, we have applied it to non-linear PDEs which are homogeneous as well as non-homogeneous.

Example 4.1: Consider the following non-linear PDE

$$\frac{\partial u}{\partial t}=u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2, \tag{4.1}$$

with initial condition

$$u(x,0)=x, \tag{4.2}$$

and

$$u(0,t)=t, u_x(0,t)=1. \tag{4.3}$$

The exact solution can be expressed as:

$$u(x,t)=x+t.$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation with respect to 't' on eqn. (4.1), we get

$$sL[u(x,t)]-u(x,0)=L\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right].$$

By using initial conditions from equation (4.2), we get

$$L[u(x,t)]=\frac{x^2}{s}+\frac{1}{s}L\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right].$$

Here, we applying the inverse Laplace transformation with respect to '*s*', on both sides:

$$u(x,t)=x^2+L^{-1}\left[\frac{1}{s}L\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]\right].$$

Now applying the DTM with respect to space variable '*x'*, we get

$$U_k(t)=\delta(k-1,t)+$$
$$L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}(r+2)(r+1)U_{r+2}(t)U_{k-r}(t)+\sum_{r=0}^{k}(r+1)(k-r+1)U_{r+1}(t)U_{k-r+1}(t)\right]\right], \tag{4.4}$$

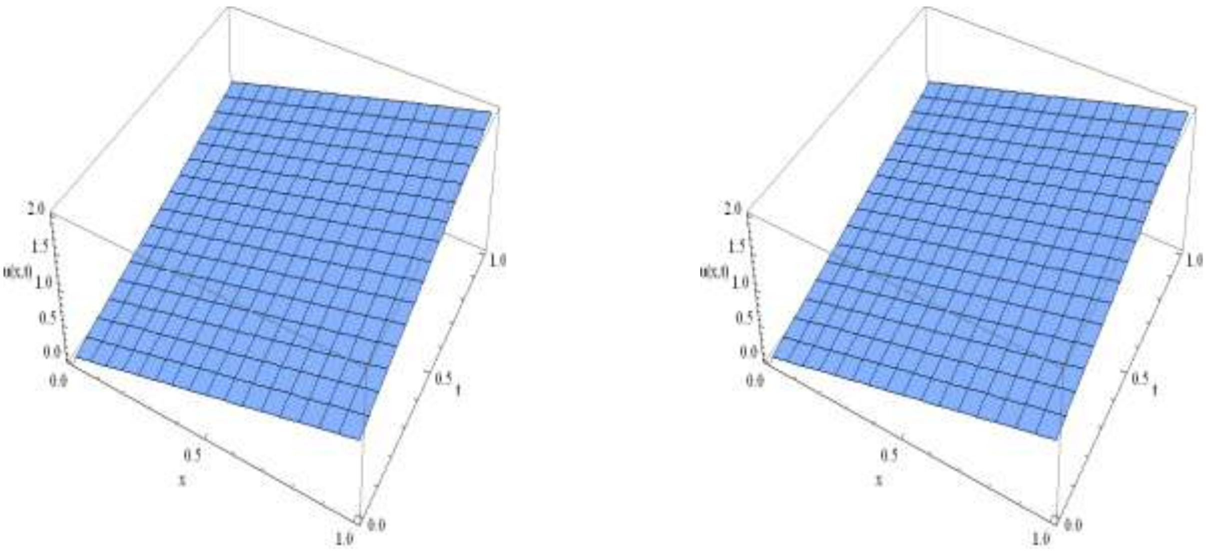
from the initial condition given by Eq. (4.3), we have

$$U_0(t)=t,\ U_1(t)=1. \tag{4.5}$$

Substituting (4.5) into (4.4) and by straightforward iterative steps and we get the component $U_k(t)$, $k\geq 0$ of the DTM can be obtained. When we substitute all values of $U_k(t)$ into equation (3.11), then the series solution can be formed as

$$u(x,t)=x+t.$$

which is the exact solution.



(a) **(b)**

Fig. 1: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.1.

Example 4.2: Consider the following non-linear PDE

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} - u^2 + e^x, \tag{4.6}$$

with initial condition

$$u(x,0) = 0, \tag{4.7}$$

and

$$u(0,t) = t, u_x(0,t) = t. \tag{4.8}$$

The exact solution can be expressed as:

$$u(x,t) = e^x t.$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation with respect to ‘t’ on eqn. (4.6), we get

$$sL[u(x,t)] - u(x,0) = L\left[u\frac{\partial^2 u}{\partial x^2} - u^2 + e^x\right].$$

By using initial conditions from equation (4.7), we get

$$L[u(x,t)] = \frac{1}{s}L\left[u\frac{\partial^2 u}{\partial x^2} - u^2 + e^x\right].$$

Here, we applying the inverse Laplace transformation with respect to ‘*s*’, on both sides:

$$u(x,t) = L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - u^2 + e^x\right]\right].$$

Now applying the DTM with respect to space variable ‘*x’*, we get

$$U_k(t) = L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}(r+2)(r+1)U_{r+2}(t)U_{k-r}(t) - \sum_{r=0}^{k}U_r(t)U_{k-r}(t) + \frac{1}{k!}\right]\right], \tag{4.9}$$

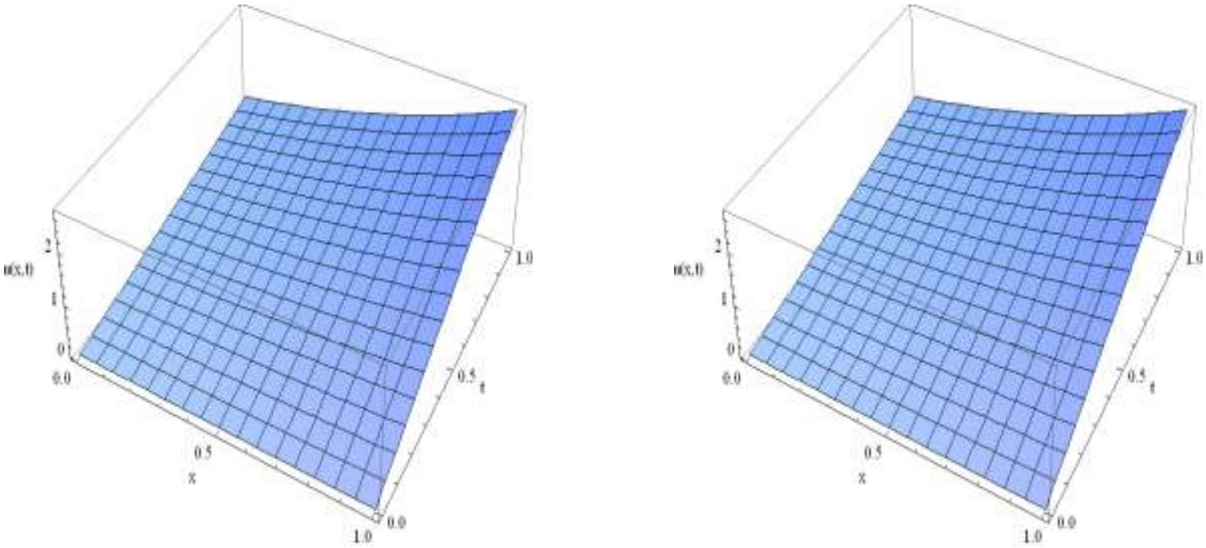
from the initial condition given by Eq. (4.8), we have

$$U_0(t) = t,\ U_1(t) = t. \tag{4.10}$$

Substituting (4.10) into (4.9) and by straightforward iterative steps and we get the component $U_k(t),\ k \geq 0$ of the DTM can be obtained. When we substitute all values of $U_k(t)$ into equation (3.11), then the series solution can be formed as

$u(x,t) = e^x t.$

which is the exact solution.



(a) **(b)**

Fig. 2: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.2.

Example 4.3: Consider the following non-linear PDE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = 2x^2 - 2t^2 + x^4 t^4, \tag{4.11}$$

subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \tag{4.12}$$

and the Neumann boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) = 2t^2. \tag{4.13}$$

According to the Laplace differential transform method, firstly we applying the Laplace transformation with respect to ‘t’ on equation (4.11), we get

$$s^2 L[u(x,t)] - su(x,0) - u_t(x,0) = \frac{2x^2}{s} - \frac{4}{s^3} + \frac{4!\,x^4}{s^5} + L\left[\frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t)\right].$$

By using initial conditions from equation (4.12), we get

$$L[u(x,t)] = \frac{2x^2}{s^3} - \frac{4}{s^5} + \frac{4!\,x^4}{s^7} + \frac{1}{s^2} L\left[\frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t)\right].$$

Here, we applying the Inverse Laplace transformation w.r.t. ‘s’ on both sides, and we get

$$u(x,t) = x^2 t^2 - \frac{t^4}{6} + \frac{x^4 t^6}{30} + L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t)\right]\right].$$

Now applying the Differential transformation method with respect to space variable ‘x’, we get

$$\begin{aligned} U_k(t) &= t^2 \delta(k-2) - \frac{t^4}{6}\delta(k) + \frac{t^6}{30}\delta(k-4) \\ &+ L^{-1}\left[\frac{1}{s^2} L\left[(k+2)(k+1)U_{k+2}(t) - \sum_{r=0}^{k} U_r(t) U_{k-r}(t)\right]\right], \end{aligned} \tag{4.14}$$

From the initial condition given by Eq. (4.13), we have

$$U_0(t) = 0. \tag{4.15}$$

And let us assume

$$U_1(t) = at^2. \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.14) and by straightforward iterative steps, we obtain

$$U_2(t) = t^2,\ U_3(t) = \frac{a}{3},\ U_4(t) = \frac{a^2t^4}{12},\ \ldots \tag{4.17}$$

From equation (3.10), we take

$$u_x(1,t) = \sum_{i=0}^{\infty} iU_i(t) = 2t^2\ ,$$

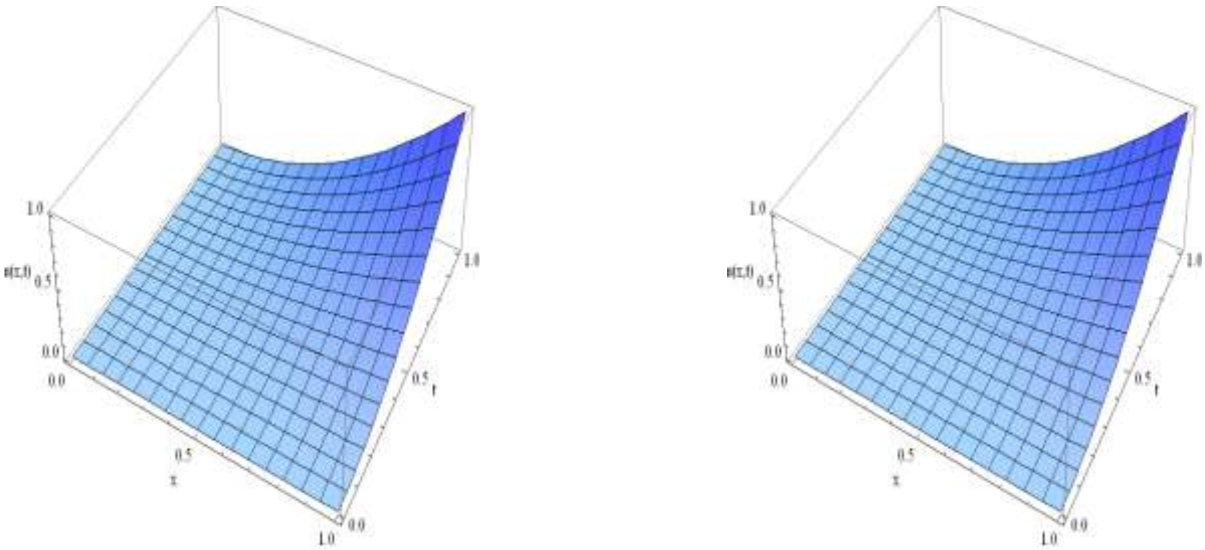
and, we get

$$a = 0.$$

Substituting the value of 'a' into equations (4.16) and (4.17), now by straightforward iterative steps, we get the component $U_k(t),\ k \geq 0$ of the LDTM can be obtained. When we substitute all values of $U_k(t)$ into equation (3.11), then the series solution can be formed as

$$u(x,t) = x^2t^2.$$

which is the exact solution.



(a) **(b)**

Fig. 3: The behavior of the (a) Exact solution, (b) LDTM solution, w.r.to x and t are obtained for the Example 4.3.

V. Conclusion

In this paper, we applied LDTM to find the exact solution of homogeneous and non-homogeneous non-linear PDEs with initial or boundary conditions. The aim of this paper is to describe that the LDTM is so powerful and efficient which gives approximations of high accuracy and closed form solutions. It gives more reliable and reasonable solution for non-linear initial and boundary value problems. The result shows that LDTM is powerful mathematical tool for solving nonlinear PDEs. The LDTM solution can be calculated easily in short time and the graphs were performed by using mathematica-8.

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